STUDENT'S CONCEPT OF INFINITY IN THE CONTEXT OF A SIMPLE GEOMETRICAL CONSTRUCT

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The research described in this paper was undertaken to determine student-teachers' understanding of infinity in a geometrical context. The methods of analysis of students' responses is presented and these were found to be universally applicable. The findings show that school mathematics does not generally develop the students' ideas of infinity (Eisenmann, 2002). We believe that discussion about infinity could lead to the development of cognitive ability and hence the need for teachers to have a sound knowledge of infinity and the necessary communicative skills.

INTRODUCTION AND FRAMEWORK

Infinity has always intrigued mathematicians, philosophers and other scientists throughout our history. It is such a profound concept that consideration of it has always led thinkers to formulate the most deep and innovatory ideas of their time (Bolzano, Newton, Leibniz, Cantor). Early thoughts about it had to wait until mathematics changed from a purely practical discipline to a more intellectual one, around 600 BC. However the.Greeks did not develop their ideas. Zeno and Archimedes were exceptions (Hejny, 1978). It was not until the sixteenth century that further developments were made. John Wallis, the first person to have used the symbol ∞, in 1650 discovered a formula which used the division of two 'infinite products'(Maor, 1991).

Early in its development, infinity was considered in relation to either the very small or very large and considered generally as potential. Cantor changed these ideas completely by accepting actual infinity as a mathematical entity and the infinite set as a totality (Cantor, 1955).

Cantor's theories and the concept of infinity, especially actual infinity, are still found difficult to grasp today by many students of mathematics. These difficulties result in vague and inconsistent answers to be given to problems involving infinite sets, for example. They have been researched amongst others by Fischbein, Tall, Tirosh and Tsamir (1979, 1980, 1996, 1999) mainly in a numerical setting and the research looked at the influence of students' intuition of finiteness on their ideas of infinity. Fischbein, Tirosh and Hess (1979) found that pupils considered infinity to be either a process which was infinite or one which came to an end.

Monaghan (1986, 2001) argues that when ideas about limiting processes are presented in a geometric context they are stronger than when presented in a numeric context. It is assumed that basic geometrical objects can be visualised and are considered part of the real world hence it is more difficult to consider the infinite phenomena of these objects. Most of current mathematical education researches into infinity have been carried out in the arithmetical field. We have used the geometrical field for our research and we hope that our results will contribute to the knowledge of a student's understanding of infinity and to the methods of gaining insight into it.

AIMS OF RESEARCH

The word infinity belongs to children's natural vocabulary from their early years. It is often linked to two emotions; excitement of penetrating into something beyond the real world, and fear of the unknown. The children cannot have any direct experience of infinity from the real world so they project their real experiences into the word 'infinity'. Hence by a mental process using abstraction, absolutism and idealisation they create a mental construct, a tacit model of infinity (Fischbein, 2001).

We believe that, as in history, the investigation of the phenomenon of infinity considerably enhanced the development of human knowledge, so consideration of infinity by students could significantly contribute to their cognitive development. To discuss the phenomena of infinity with students demands deep preparation by the teacher. The teacher should have a sound knowledge of infinity him/herself but also have knowledge of the way students perceive infinity. This second aspect requires the teachers to undertake small scale research in their classrooms so that they able to recognise and rectify the pupil's own misconceptions and to distinguish whether the underlying problem was with their understanding of infinity or with their inability to express their ideas correctly.

There are ample topics in the school mathematics curriculum which offer good opportunities for discussions on infinity such as natural and rational numbers, progressions, limits, line segments, straight line etc., however there is no curriculum heading of infinity in school syllabuses as far as we are aware. Most teachers only superficially refer to infinity occasionally by saying 'the series of natural numbers goes on and on into infinity', 'the plane can be extended in all directions into infinity', etc. and do not go deeper into this phenomenon because he/she would not wish to show their own uncertainty. It therefore depends entirely on the teacher whether or not they can lead effective discussions.

Our aim for the research was to find out what the student's understanding of infinity was when related to a geometrical context. We hoped the research would give us an insight into the cognitive mechanisms which determine the development of the understanding of the concept of infinity, to identify obstacles which hinder such a development and find tasks and procedures which will help to educate and re-educate students to avoid or overcome the obstacles. A secondary aim was to develop a universal methodology for this research which could be easily used by practising teachers.

METHODOLOGY

The research was carried out in two stages. In 1995 the methodology for this research was developed by a team led by M. Hejny at Charles University, Prague (see Jirotkova, 1996, 1997, 1999). During this stage, particular emphasis was placed on the responses of the students to Task 1. The second stage was to verify the method of analysis of the responses and then to extend the research to the individual by interviewing selected students and using the series of tasks developed for this purpose.

Research Tool

We did not want the students to be aware that we were particularly seeking information about their understanding of infinity so our research tool was one which contained an indirect request for this. The students mentioned infinity spontaneously. They were given:

<u>Task 1</u>: Try to define your own understanding of the concept of a straight line.

Our analysis in the first stage of our research resulted in the development of a series of tasks to be used with individual students for diagnosis, follow-up work and the development of their thinking and communicative skills. We list below two of them which are referred to in our analysis.

<u>Task 2</u>: Look at the statements below. Decide which of the two children is correct.

Adam: *A straight line has two 'infinities'. If I go in one direction I'll reach infinity. If I go in the opposite direction I'll also reach infinity.*

Boris: *Those two 'infinities' are the same, so there is only one infinity on a straight line. It is the place where both ends join together like a circle.*

<u>Task 3</u>: Given a straight line *b* and a point *A* not lying on b, consider all squares *ABCD* whose vertex *B* is on the straight line *b*. Draw square *ABCD* with: (a) the smallest possible area, (b) the greatest possible area. Draw the diagonals *AC* and BD and mark the centre of the square. If you do not have enough room on your paper to mark a certain point draw arrows to indicate the direction in which it lies.

Research Sample

In 1995 Task 1 was given to 72 primary school student teachers. They were in their first year at University and had not taken a course in geometry nor had any course influenced their understanding of infinity. Hence their knowledge of infinity was that gained at school or through life experiences which probably developed tacit models of infinity in their minds (Fischbein, 2001). In 2002 the same tool (Task 1) was used with 102 student teachers: 43 first year students studying to teach mathematics in secondary schools and in the process of having their first course on geometry; 25 students studying to teach special needs pupils who had not taken any University course in geometry; 10 second year primary education students who had taken a course in geometry; 24 first year primary education students who had not taken a course in geometry at the university.

The same task was given to 18 English students who were studying on a primary post-graduate certificate of education course. This was done because the translation of the responses of the Czech students into English might lose or cause slight changes in emphasis from the original responses. We hoped to gain authentic English statements with which to compare the translations.

Method of Research

Task 1 was given to all the students verbally. They were asked to write their responses on paper. No time limit was set. Each of the 174 **responses** received were considered and from them we chose those responses which contained the explicit use of the word 'infinity', 'infinite', 'end', 'endless', 'never-ending', 'end-point'…. We then split these responses into simple ideas which we called **statements.** The statements which did not

mention words similar to those listed above were discarded. In this way we were left with 92 different statements. For instance in Alice's response: *A straight line is a line segment of infinite length, (1) and is a simple direct line which does not have and end or a beginning (2) or both are in infinity (3). It could be defined as a circle of infinite radius (4).* Alice's response gave us four contributions to our list of statements. The statements, which we considered had similar meanings, were grouped together and represented by a single authentic statement, which we called a **phrase**. For instance the authentic phrase 'an infinite set of points' represented several other statements: *join of infinitely many points, non finite set of points ordered linearly, consists of infinitely many points etc*. In this way we got 26 different phrases.

Contrary to the classification of students by their understanding of infinity used by Sierpinska (Tall, 1996), we have classified the phrases used by the students. That is, we did not analyse individual student's understanding, just the phrases within their responses. This was the first level of our analysis.

In the second level we decided to classify the phrases in a non-mathematical way, that is, we grouped them by the grammatical aspect. In **Group A** we put all phrases in which infinity was expressed as a noun as if the author accepted the existence of infinity. In **Group B**, we put those phrases which expressed infinity as an adjective or adverb. In these phrases the existence of infinity was not indicated directly and was considered to be a property of the straight line. This property is defined in the phrases of **Group C** implicitly. The students formulate it as the opposite or absence of finiteness by denying the existence of the end(s), end-points of a straight line.

In level 3 of the analysis we looked for those phenomena which created the students' understanding of infinity, described them and classified the students' responses according to them. We consciously interpreted the phrases in a way which exceeded the preciseness of the author of the phrase. We were aware that the students' images of the concept of infinity might be vague and fuzzy and that they may also lack the ability to articulate their images. Our experience would indicate that such an approach gives an insight into the whole problem. It also enabled us to suggest ideas of how to diagnose a student's difficulties and find means of developing their understanding and communicative ability.

These levels of analysis took place in both stages of the research. In the second stage of our research as indicated in the aims above, we went further and focussed on the student as an individual. When a response from a student contained statements which were contradictory, or were inconsistent, we interviewed the student to see whether the contradictions or inconsistencies were caused by misconceptions of the word infinity or the lack of communicative skills. For this we chose some of the series of suggested tasks mentioned above.

RESULTS AND DISCUSSION

Results

The first level of our analysis resulted in the following list of phrases classified into three groups. All phrases refer to the straight line.

Group A

A1…..begins and ends at infinity,
A2…..has its beginning and ending at infinity,
A3…..the end points are at infinity,
A4…..beginning at infinity, going on in the opposite direction to infinity,
A5…..goes to infinity in both directions,
A6…..starts at infinity and leads to infinity,
A7…..goes from infinity to infinity,
A8…..could be extended into infinity.

Group B

B9…..has an infinite number of points,
B10…is infinite,
B11…is an infinite connection,
B12…is an infinite line segment,
B13…is infinitely long,
B14…is an infinite figure of points,
B15…is an infinite set of points,
B16…is an infinite series of points.

Group C

C17…does not have either beginning or end point,
C18…without beginning or end,
C19…does not begin and end anywhere,
C20…with the beginning and end missing,
C21…with unlimited beginning and end,
C22…I can never see the end or the beginning,
C23…does not end
C24…not ending anywhere
C25…is not finite, not ended,
C26…it is not possible to determine the end point.

Returning to the example of Alice above, statements (1) and (4) were classified in phrases B12 and B14 respectively, statement (2) in C17 and statement (3) in A3.

Group A. In this group's phrases we found four polar phenomena which characterised the students' ways of expressing infinity or the infiniteness of a straight line.

P1 - number of 'infinites' on the straight line, one or two;
P2 - number of times the word 'infinity' was used in a phrase, one or two;
P3 - quality of infinity: (a) beginning or end, (b) locality or direction;
P4 - potentiality or actuality.

DISCUSSION

We now consider each of these phenomena more closely:

P1. Phrase A8 speaks of one infinity, whereas all the other phrases could imply that there were two infinities. However we cannot exclude the possibility that both the infinities implied by the writers were the same in their imagination. This uncertainty led us to construct Task 2 (see above). In this Boris explicitly declares '...there is only one infinity on a straight line. It is the place where both ends join together, like a circle'. This idea corresponds to the idea of a straight line in topology.

P2. Phrases A1, A2, A3, A5, and A8 use the word ‘infinity’ once and in A4, A6 and A7 it is used twice. From our comments related to the phenomenon P1 you can see it is not possible to say that the number of times the word infinity is used determines the number of infinities in a student’s understanding.

P3(a). When the two infinities are mentioned we can consider the different qualities of the infinities. In phrases A1, A2, A6 the quality of the infinities are different, one is the beginning and the other, the end. In the others the quality is the same.

P3(b). When it is said that the straight line has its end in infinity, infinity is being used as a label for a particular place on the straight line which is ‘very far away’, ‘unreachable’. These responses imply that the authors of them might not have considered the infiniteness of the straight line. On the other hand, when they say the straight line ‘goes towards infinity’ (A4, A5, A7) they speak about infinity as a direction and that the straight line is like a ‘signpost’ pointing in the direction of infinity. In A4 and A7 the word infinity is used twice and in each case the quality of it is different. The first time the word is used, in A4, it signifies a place where the process of creating the line starts. The second ‘infinity’ is a signpost for the direction in which the straight line goes. In A7 these infinity qualities are reversed ‘goes from infinity to infinity’. In our interpretation the other phrases refer to infinity as a location. We are aware that our interpretations depend on our own experiences so again we created Task 3 to help us determine how the students understood the aspect of quality.

P4. If you compare A2 and A5 then phrase A5 can be interpreted as stressing the process of creating the straight line. The writer considered the process and not the completion of it. The straight line thus existed in its possibility (potentiality) of being realised and not in its completion. In this case we interpreted infinity as having two properties: an indication of direction and that it was unreachable. Such an interpretation of infinity we called potential. Phrase A2 speaks about the beginning and end of the straight line as if they were two points at some place called infinity. The writers of such statements looked at the line as a whole, an object, which has been completed. The image of infinity as a fixed locality and its infiniteness supports this understanding and is close to what is called in mathematics actual infinity. This was also found in phrases A1 and A3 ‘begins and ends/end points are at infinity’.

<u>Group B</u>. In these statements, infinity is considered to be a property of the straight line but does not state the existence of infinity directly. The analysis of group B was based on the gradual elimination of single phrases. We started to look for unique phenomena within the phrases but most were applicable only to part of group B. After ordering the phrases the analysis enabled us to tabulate it as follows:

<table>
<tr><td rowspan="5">has</td><td colspan="9">is</td></tr>
<tr><td rowspan="4">directly</td><td colspan="8">indirectly</td></tr>
<tr><td colspan="2" rowspan="3">quality</td><td colspan="6">quantity</td></tr>
<tr><td colspan="3" rowspan="2">continuously</td><td colspan="3">discretely</td></tr>
<tr><td colspan="2">without order</td><td>with order</td></tr>
<tr><td>B9</td><td>B10</td><td>B11</td><td colspan="2">B12</td><td>B13</td><td colspan="2">B14</td><td>B15</td><td>B16</td></tr>
</table>

The table shows how it is possible to bifurcate the phrases progressively. Infinity was considered as belonging to the straight line by the use of 'has' in B9, in all the other phrases it was considered as a property of the straight line using 'is'. This was the first bifurcation. For the second, the infinity property was expressed directly in B10 but indirectly in the others. The connection between infinity and a straight line was quality in B11 and quantity in the others. Phrase B12 was difficult to classify since it could have been interpreted in both ways. The quantity property is represented by an object which is continuous in B13 and discrete in the remainder. B14 caused similar problems to B12 since it could be interpreted as either continuous or discrete. Note that discreteness can be without order B15 or with order B16.

We considered that the students we investigated understood the word 'figure' as referring to a continuous object hence we felt that B14 should be classified with B13 rather than B15.We used the table as a tool for the analysis of Group B. We accept it is not universal because it relates only to those phrases taken from our sample. This means that our analysis of Group B is different from that of Group A where the criteria are universal. Nevertheless the methodology in which the table was created, that is the division into a series of phenomena by which we characterised single cells, is universal.

Group C. As in group B the authors did not speak of infinity directly. They also deny the existence of the end(s) of the line. We were aware though that saying the end does not exist did not imply that infinity is being inferred. However we thought that in the phrases C17 to C22, the non-existence of the beginning and end was connected with the image of the infiniteness of the straight line in the sense of them being unattainable. In C22 and C26 the existence of the end is declared but the end is moved beyond the writer's horizon, which from our perspective does not influence the structure of the straight line.

In the second stage of our research we interviewed those students who gave contradictory or inconsistent statements and presented them with tasks to help them and us to clarify their understanding of infinity and the means to express their ideas. The results of this work will be the subject of a further paper.

CONCLUSIONS AND FURTHER RESEARCH

We confirmed that the original method of analysis was sound and provided a useful tool to determine students' concepts of infinity. The development of the supplementary tasks for diagnosis and follow-up work were found to be particularly useful. We have listed two of these to which we refer in our analysis. Student's reactions to them confirmed our initial interpretations. The research is currently continuing with particular emphasis on student's solutions of the supplementary tasks, which will allow us to compare their understanding of infinitesimally small and infinitely large infinity. We are working on methods of analysis to do this.

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